

On the divisibility of odd perfect numbers by a high power of a prime, II^{*†}

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Abstract

We shall give an explicit upper bound for the smallest prime factor of multiperfect numbers of the form $N = p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t}$ with β_1, \dots, β_t bounded by a given constant.

1 Introduction

Let $\sigma(N)$ denote the sum of divisors of N for a positive integer N and define $h(N) = \sigma(N)/N$. An integer N is called to be perfect if $h(N) = 2$. It is one of oldest and most infamous problems whether there exists any odd perfect number. Moreover, it is also unknown whether there exists any odd integer N with $h(N) = k$ for some integer $k > 1$.

Although it is unknown that whether there exists any odd perfect number, it is known that an odd perfect number must satisfy various conditions. Suppose that N is an odd perfect number. Euler has shown that $N = p^\alpha q_1^{\beta_1} \dots q_t^{\beta_t}$, where p, q_1, \dots, q_t are distinct odd primes with $p \equiv \alpha \equiv 1 \pmod{4}$ and β_1, \dots, β_t even. Steuerwald [18] proved that we cannot have $\beta_1 = \dots = \beta_t = 2$. If $\beta_1 = \dots = \beta_t = \beta$, then it is known that $\beta \neq 4$ (Kanold [11]), $\beta \neq 6$ (Hagis and McDaniel [8]), $\beta \neq 10, 24, 34, 48, 124$ (McDaniel and Hagis [15]), $\beta \neq 12, 16, 22, 28, 36$ (Cohen and Williams [3]). In their paper [9], Hagis and McDaniel conjecture that $\beta_1 = \dots = \beta_t = \beta$ does not occur. The author [19] proved that there are only finitely many odd perfect numbers for any given β . McDaniel [13] proved that we cannot have $\beta_1 \equiv \dots \equiv \beta_t \equiv 2 \pmod{6}$, i.e., 3 cannot divide all of $\beta_1 + 1, \beta_2 + 1, \dots, \beta_t + 1$. If m divides all of $\beta_1 + 1, \beta_2 + 1, \dots, \beta_t + 1$, then it is known that $m \neq 35$ (Hagis and McDaniel[9]) and $m \neq 65$ (Evans and Pearlman[4]).

However, if we relax the condition that there exists some integer dividing all of $\beta_1 + 1, \beta_2 + 1, \dots, \beta_t + 1$, then the situation becomes quite different. The simplest problem in this direction would be whether there exists an odd perfect number of the form $p^\alpha q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t}$ with $p \equiv \alpha \equiv 1 \pmod{4}$ and $\beta_i \leq 4$. This problem has been studied by McDaniel [14], Cohen [1]. These papers give *lower*

*2010 Mathematics Subject Classification: 11A25, 11A51, 11N36.

†Key words and phrases: Odd perfect numbers; multiperfect numbers; sieve methods.

bounds for the smallest prime factor of N : the first paper shows it must be ≥ 101 , the second shows it must be ≥ 739 .

In general, we can make a conjecture that for a fixed finite set \mathcal{P} of integers, a fixed rational number n/d and a fixed integer s , there exist only finitely many odd n/d -perfect numbers $N = p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t}$ with β_1, \dots, β_t contained in \mathcal{P} .

This conjecture still seems to be far beyond reach, though this conjecture is weaker than the finiteness (or non-existence) conjecture of odd n/d -perfect numbers. In the preprint [20], using sieve methods, the author has proved that for an fixed finite set \mathcal{P} of integers, a fixed rational number n/d and a fixed integer s , there exists an effective constant C such that odd n/d -perfect numbers of the form $N = p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t}$ with β_1, \dots, β_t contained in \mathcal{P} must have a prime divisor smaller than C . Moreover, the author has proved that, in the case N is perfect and $\beta_i \leq 4$, then C can be taken to be $\exp(4.97401 \times 10^{10})$.

Using the author's method, but with the aid of the large sieve instead of Selberg's sieve used by the author [19], Fletcher, Nielsen and Ochem[5] proved that, if $N = p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t}$ satisfies $h(N) = n/d$ and for each i , $\beta_i + 1$ has a prime factor belonging to a finite set \mathcal{P} of primes, then N has a prime divisor small than a effective constant C , depending only on n, s and \mathcal{P} . Moreover, they proved that the smallest prime factor of an odd perfect number N satisfying the above condition with $\mathcal{P} = \{3, 5\}$ lies between 10^8 and 10^{1000} , improving results in [1] and [20].

However, they did not give an explicit value for C in other cases. In this paper, the author would like to give an explicit upper bound for C in general cases.

Theorem 1.1. *Let \mathcal{P} be a finite set of primes and $n, d, \beta_1, \dots, \beta_t$ be positive integers such that for any $i = 1, \dots, t$, $\beta_i + 1$ is divisible by at least one prime in a set \mathcal{P} and let P denote the product $\prod_{p \in \mathcal{P}} p$. Define $\Omega_{\mathcal{P}}(x)$ to be the number of prime factors of x which belongs to \mathcal{P} , counting multiplicity. Furthermore, let*

$$\begin{aligned} s_0 &= s + \omega(n) + \Omega_{\mathcal{P}}(n), \kappa = \frac{l-1}{\varphi(P)}, \\ x_1 &= x_1(l) = \max\{\exp \max\{l, \exp 13.3\}, 10s_0(l-1) + 1\}, \\ x_2 &= x_2(l) = \exp \max\{101 \log x_1, \log^{\frac{1}{1-\kappa}} x_1, P, \exp 13.3\} \end{aligned} \quad (1)$$

and C_0 be the maximum among quantities $2(d+1)s, x_2 = x_2(l)$ and

$$\exp \frac{(63.2\varphi(P) + 598.6(l-1))\#\mathcal{P} \log x_1}{(l-1) \log \frac{n}{d}} \quad (2)$$

with l running over all primes in \mathcal{P} .

If $N = p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t}$ satisfies $h(N) = \frac{n}{d}$, then N has a prime factor smaller than C_0 .

It would be notable that our upper bound is the order of exponential of $\varphi(P) \max \mathcal{P} |\mathcal{P}|$, rather than double-exponential of $\varphi(P) \log P$ as in Theorem 3 of [5].

It would also be notable that no absolute upper bound is known for the smallest prime factor of a *general* odd perfect number if it exists at all; another known result is Grün's result[7] that the smallest prime factor must be smaller than $\frac{2}{3}\omega(N) + 2$, where $\omega(N)$ denotes the number of distinct prime factors of N .

2 Distribution of primes in arithmetic progressions

In order to make our upper bound explicit, we need some explicit results on primes in arithmetic progressions.

Chen and Wang [2] proved that if $x \geq \exp \exp 9.7$, $k \leq \log^3 x$ and χ is a Dirichlet character modulo k , then

$$|\psi(x, \chi)| \leq \frac{0.077x}{\log^{10.35} x} + E \frac{x^\beta}{\beta}, \quad (3)$$

where β denotes a real zero of χ greater than $1 - 0.1077/\log k$ and $E = 1$ if it exists (For more general results, see the author's recent paper [21]).

Since Theorem 3 of [12] gives that $\beta \leq 1 - \pi/0.4923k^{1/2} \log^2 k$, we see that if $k \leq \log x$ and $x \geq \exp \exp 13.3$, then

$$\left| \psi(x; k, a) - \frac{x}{\varphi(k)} \right| \leq \frac{0.305x}{\varphi(k) \log x}. \quad (4)$$

In other words, putting $x_0 = \exp \max\{k, \exp 13.3\}$, the inequality (4) holds for $x \geq x_0$.

Based on this inequality, we shall prove the following estimates.

Lemma 2.1. *Let w, z be an arbitrary real number with $z \geq w \geq x_0$. Then the inequality*

$$\sum_{\substack{w < p \leq z, \\ r \equiv 1 \pmod{k}}} \frac{\log p}{p} < \frac{1}{\varphi(k)} \left(\log \frac{z}{w} + \frac{0.31}{\log^2 w} + \frac{0.31}{\log^2 z} + \frac{0.31}{\log w} \right) \quad (5)$$

and

$$\sum_{\substack{w < p \leq z, \\ r \equiv 1 \pmod{k}}} \frac{1}{p} > \frac{\log \log z - \log \log w}{\varphi(k)} - \frac{0.78}{\varphi(k) \log^2 w} \quad (6)$$

holds.

Moreover, if $z \geq x_0^{101}$, then we have

$$\sum_{\substack{p \leq z, \\ p \equiv 1 \pmod{k}}} \frac{\log p}{p} < \frac{1.01}{\varphi(k)} \log z \quad (7)$$

and

$$\prod_{p < z} \left(1 - \frac{1}{p}\right) < \left(\frac{\log z}{\log x_0}\right)^{1/\varphi(k)} \exp\left(-\frac{0.78}{\varphi(k) \log^2 x_0}\right). \quad (8)$$

Proof. We begin by noting that (4) yields

$$\left|\theta(x, k, 1) - \frac{x}{\varphi(k)}\right| < \frac{0.31}{\varphi(k) \log^2 x} \quad (9)$$

for $x \geq x_0$.

Now we shall prove (6) and (8). By partial summation, we have

$$\begin{aligned} & \sum_{\substack{w < p < z, p \equiv 1 \pmod{k}}} \frac{1}{p} \\ & > \frac{\log \frac{\log z}{\log w}}{\varphi(k)} - \frac{0.31}{\varphi(k)} \left(\frac{1}{\log^2 z} + \frac{1}{\log^2 w} + \int_w^z \frac{(1 + \log t) dt}{t \log^4 t} \right) \\ & > \frac{\log \frac{\log z}{\log w}}{\varphi(k)} - \frac{0.31}{\varphi(k)} \left(\frac{1}{\log^2 w} + \frac{1}{\log^2 z} + \frac{1}{2 \log^2 w} + \frac{1}{3 \log^3 w} \right) \\ & > \frac{\log \frac{\log z}{\log w}}{\varphi(k)} - \frac{0.78}{\varphi(k) \log^2 w} \end{aligned} \quad (10)$$

for $z > w \geq x_0$ and obtain (6). In particular, we have

$$\sum_{\substack{p < z, p \equiv 1 \pmod{k}}} \frac{1}{p} > \sum_{\substack{x_0 < p < z, p \equiv 1 \pmod{k}}} \frac{1}{p} > \frac{\log \log z - \log \log x_0}{\varphi(k)} - \frac{0.78}{\varphi(k) \log^2 x_0} \quad (11)$$

and therefore

$$\begin{aligned} \prod_{\substack{p < z, p \equiv 1 \pmod{k}}} \left(1 - \frac{1}{p}\right)^{-1} &= \exp \sum_{\substack{p < z, p \equiv 1 \pmod{k}}} \left(\frac{1}{p} + \frac{1}{2p^2} + \cdots\right) \\ &> \exp \sum_{\substack{p < z, p \equiv 1 \pmod{k}}} \frac{1}{p} \\ &> \left(\frac{\log z}{\log x_0}\right)^{1/\varphi(k)} \exp\left(-\frac{0.78}{\varphi(k) \log^2 x_0}\right) \end{aligned} \quad (12)$$

for $z \geq x_0$, which gives (8).

Next, we shall prove (7). Similarly to above, partial summation gives

$$\begin{aligned} \sum_{\substack{p < z, p \equiv 1 \pmod{k}}} \frac{\log p}{p} &= \frac{\theta(z; k, 1)}{z} + \int_{2k+1}^z \frac{\theta(z; k, 1)}{t^2} dt \\ &< \frac{1}{\varphi(k)} \left(1 + \frac{0.31}{\log^2 z}\right) + \int_{2k+1}^z \frac{\theta(z; k, 1)}{t^2} dt. \end{aligned} \quad (13)$$

By the Brun-Titchmarsh theorem given in [16], we have

$$\begin{aligned} \int_{2k+1}^{x_0} \frac{\theta(z; k, 1)}{t^2} dt &< \frac{2}{\varphi(k)} \int_{2k}^{x_0} \frac{\log t dt}{t \log \frac{t}{k}} \\ &= \frac{2}{\varphi(k)} \left(\log \frac{x_0}{2k} + \log k \left(\log \frac{\log x_0}{\log k} - \log \log 2 \right) \right) \\ &< \frac{2.0015}{\varphi(k)} \log x_0. \end{aligned} \quad (14)$$

We can easily see that (9) gives

$$\int_{x_0}^z \frac{\theta(z; k, 1)}{t^2} dt < \frac{1}{\varphi(k)} \left(\log \frac{z}{x_0} + \frac{0.31}{\log x_0} \right). \quad (15)$$

Inserting these upper bounds into (13) yields

$$\sum_{p < z, p \equiv 1 \pmod{k}} \frac{\log p}{p} < \frac{1}{\varphi(k)} (\log z + 1.0016 \log x_0) < \frac{1.01 \log z}{\varphi(k)} \quad (16)$$

for $z \geq x_0^{101}$, giving (7).

Finally, (5) immediately follows by using the partial summation

$$\sum_{w \leq p < z, p \equiv 1 \pmod{k}} \frac{\log p}{p} = \frac{\theta(z; k, 1)}{z} - \frac{\theta(w; k, 1)}{w} + \int_w^z \frac{\theta(z; k, 1)}{t^2} dt \quad (17)$$

and (9). \square

3 Upper bound sieve

Another result that we need is a standard result in large sieve theory. However, for convenience to compute explicit bounds, we must use an explicit (but a little sophisticated) upper bound sieve formula. There are several explicit upper bound sieve formulae to obtain explicit upper bound for the implied constant in an upper bound sieve. In [20], the author used the upper bound formula following from Selberg's sieve. But here we shall use the large sieve formula used by Fletcher, Nielsen and Ochem[5], which enabled them to obtain a considerably stronger estimate than in the author's paper [20].

Firstly, we would like to introduce some notations. Let A and Ω_p , where p is an arbitrary prime number, be sets of residue classes \pmod{p} , B be a positive integer, X be a real number, and $\rho(n)$ be a multiplicative arithmetic function satisfying $\rho(p) = |\Omega_p|$ for any prime p . Denote by A_d the set of positive integers in A which belongs to Ω_p for any p dividing d . Define

$$P(z) = \prod_{p < z} p$$

$$g(p) = \frac{\rho(p)}{p - \rho(p)}, g(d) = \prod_{p|d} g(p),$$

$$V(Q) = \prod_{p|Q} \left(1 - \frac{\rho(p)}{p}\right)$$

and

$$G_z(u) = \sum_{d \leq u, d|P(z)} g(d),$$

where p runs over primes. Finally, we define $S(A, P) = S(A, P, \Omega)$ to be the number of integers in A which does not belong to Ω_p for any prime p dividing P .

Now we introduce two lemmas concerning the large sieve inequality. These inequalities allow us to calculate an upper bound in Theorem 1.1 explicitly.

Lemma 3.1. *It holds for any $u \geq 1$ that*

$$S(A, P(u)) \leq \frac{X + u^2}{G_u(u)}. \quad (18)$$

Proof. This is Theorem 7.14 in p.p.180–181 in [10]. \square

Lemma 3.2. *Let us denote s*

$$B(z) = \frac{1}{\log z} \sum_{p < z} \frac{\rho(p) \log p}{p} \quad (19)$$

and

$$\psi(K, t) = \max \left\{ 0, t \log \frac{t}{K} - t + K \right\} \quad (20)$$

(we believe that this ψ can easily be distinguished from the second Chebyshev prime-counting function).

If $v = (\log x)/(\log z) \geq B(z)$, then we have

$$G_z(\sqrt{x}) \geq \frac{\psi_0(v)}{V(P(z))}, \quad (21)$$

where $v = (\log x)/(\log z)$ and

$$\psi_0(v) = 1 - \exp(-\psi(B(z), v/2)). \quad (22)$$

Proof. This is Theorem 2.2.1 in p. 52 of [6] if we take $B = \sup_z B(z)$ instead of $B(z)$. But we can see that this theorem still holds with B replaced by $B(z)$. Indeed, it follows from the argument in p.p. 53–54 in [6] that

$$1 - V(P(z))G_z(u) \leq \exp \left(-c \frac{\log u}{\log z} + B(z)(e^c - 1) \right) \quad (23)$$

for any constant $c \geq 0$. Setting $u = \sqrt{x}$ and $c = \log v - \log B(z)$, we obtain the lemma. \square

4 Proof of the main result

We may assume that $P \geq 21$ by virtue of the result in [5] concerning to the case $\mathcal{P} = \{2, 4\}$ mentioned in the introduction of this paper. Let $N = p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t}$ be a solution of $h(N) = \frac{n}{d}$ and let us denote $T = \{i : q_i \equiv 1 \pmod{P}\}$. If N has a prime divisor in \mathcal{P} , then clearly N has a prime factor smaller than C_0 . We may assume without loss of generality that N has no prime divisor in \mathcal{P} and therefore $\Omega_{\mathcal{P}}(N) = 0$.

Let Q_l denote by the set of primes q_i with $\beta_i + 1$ is divisible by l . By assumption, any q_i belongs to Q_l for some l in \mathcal{P} .

Now we prove a result concerning the distribution of prime factors of N , which is the most important lemma in the proof of Theorem 1.1.

Lemma 4.1. *Let $\kappa, x_1 = x_1(l), x_2 = x_2(l)$ be the constant defined by Theorem 1.1. Moreover, let v be a real number > 4.02 and y be an arbitrary real number $\geq x_2$. If N has no prime factor $\leq y$, then we have*

$$\#\{q : q \in Q_l, q \leq x\} \leq \begin{cases} \frac{2B_1 v^2 x}{\xi(v) \log^2 x} + \sqrt{x} & \text{for } y \leq x < y^v, \\ \frac{2B_1 v^{1+\kappa} x}{\xi(v) \log^{1-\kappa} y \log^{1+\kappa} x} + \sqrt{x} & \text{for } x \geq y^v, \end{cases} \quad (24)$$

where $B_1 = e^{0.11-\gamma} \log x_1$ and $\xi(v) = 1 - \exp(-\psi(2.01, v/2))$.

Proof. Let U be the set of primes congruent to 1 (mod P) or congruent to 1 (mod l) and $\leq y$ except primes dividing nN/d .

Let r be a prime in U . Then, since $r \equiv 1 \pmod{l}$, there are $l-1$ congruent classes $g_1(r), \dots, g_{l-1}(r) \pmod{r}$ belonging to order l . Since r does not divide $\sigma(q_i^{l-1})$, q_i belongs to none of l classes $0, g_1, \dots, g_{l-1} \pmod{r}$.

Now we can apply the sieve method described in the previous section with A the set of integers $\leq x$, $X = x$, Ω_r the set of integers $\leq x$ belongs to any of congruent classes $0, g_1, \dots, g_{l-1} \pmod{r}$ for $r \in U$ and $0 \pmod{r}$ for $r \notin U$, $\rho(r) = l$ for $r \in U$ and $\rho(r) = 1$ for $r \notin U$. Thus we see that if q is a prime greater than \sqrt{x} in Q_l , then $q \in S(A, P(\sqrt{x}))$, where $P(\sqrt{x})$ denotes the product of primes in U below u . Hence we have

$$\#\{q : q \in Q_l, q \leq x\} \leq S(A, P(\sqrt{x})) + \sqrt{x}. \quad (25)$$

Observing that $G_{\sqrt{x}}(\sqrt{x}) \geq G_z(\sqrt{x})$ for $z \leq \sqrt{x}$, Lemmas 3.1 and 3.2 with $u = \sqrt{x}$ gives

$$\#\{q : q \in Q_l, q \leq x\} \leq \frac{2x}{G_{\sqrt{x}}(\sqrt{x})} + \sqrt{x} \leq \frac{2xV(P(z))}{\psi_0(v)} + \sqrt{x}, \quad (26)$$

where we put $z = x^{\frac{1}{v}}$.

Now we need to obtain an upper bound for the quantity $V(P(z))/\psi_0(v)$. There are two cases: $x \geq y^v$, i.e. $z \geq y$ and $x < y^v$, i.e. $z < y$. In both cases, we shall obtain firstly an upper bound for $B(z)$ and nextly $V(P(z))$.

We begin by considering the case $z \geq y$. By (5) and (7), observing that $y \geq x_2 \geq x_1^{101}$ and using the estimate $\sum_{p \leq z} (\log p)/p < \log z$ in [17, (3.24), p. 70], we obtain

$$\begin{aligned}
\sum_{r \leq z} \frac{\rho(r) \log r}{r} &\leq \sum_{r \leq z} \frac{\log r}{r} \\
&\quad + \sum_{\substack{r \leq y, \\ r \equiv 1 \pmod{l}}} \frac{(l-1) \log r}{r} + \sum_{\substack{y < r \leq z, \\ r \equiv 1 \pmod{P}}} \frac{(l-1) \log r}{r} \\
&\leq \log z + \log y + 1.0016 \log x_1 + \kappa(\log z - \log y) + \frac{0.32\kappa}{\log y} \\
&\leq (1 + \kappa) \log z + (1 - \kappa) \log y + 1.0016 \log x_1 + \frac{0.32\kappa}{\log y} \\
&\leq 2.01 \log z.
\end{aligned} \tag{27}$$

In other words, we have

$$B(z) < 2.01. \tag{28}$$

Nextly, we shall obtain an upper bound for $V(P(z))$. There must be at most $\Omega_{\mathcal{P}}(nN) = \Omega_{\mathcal{P}}(n)$ prime factors q_i in T since if $q_i \in T$, then $\sigma(q_i^{\beta_i})$ must be divisible by $\beta_i + 1$ and therefore by some l in \mathcal{P} . Since N is assumed to have no prime factor $\leq y$, the number of distinct prime factors of $\sigma(N) = nN/d$ congruent to 1 (mod P) or $\leq y$ is at most $s_0 = s + \omega(n) + \Omega_{\mathcal{P}}(n)$.

Thus we conclude that U consists of all primes $\equiv 1 \pmod{P}$ in T except at most s_0 primes. Hence, using (6), (8) and the well-known formula of Mertens in the form $\prod_{p < z} (1 - 1/p) < e^{-\gamma} \log^{-1} z (1 + 1/(2 \log^2 z))$ in [17, (3.26), p. 70], we have

$$\begin{aligned}
\prod_{r < z, r \in U} \left(1 - \frac{1}{r}\right) &\leq \prod_{\substack{r < z, \\ r \equiv 1 \pmod{l}, \\ r \notin U}} \frac{r}{r-1} \prod_{\substack{x_1 \leq r < z, \\ r \equiv 1 \pmod{l}}} \left(1 - \frac{1}{r}\right) \\
&\leq \left(1 + \frac{1}{x_1 - 1}\right)^{s_0} \prod_{\substack{x_1 \leq r < z, \\ r \equiv 1 \pmod{l}}} \left(1 - \frac{1}{r}\right) \\
&< \exp \frac{s_0}{x_1 - 1} \prod_{\substack{x_1 \leq r < y, \\ r \equiv 1 \pmod{l}}} \left(1 - \frac{1}{r}\right) \prod_{\substack{y \leq r < z, \\ r \equiv 1 \pmod{l}}} \left(1 - \frac{1}{r}\right) \\
&< \left(\frac{\log y}{\log x_1}\right)^{\frac{1}{l-1}} \left(\frac{\log z}{\log y}\right)^{\frac{1}{\varphi(P)}} \\
&\quad \times \exp \frac{s_0}{x_1 - 1} + \frac{0.78}{(l-1) \log^2 x_1} + \frac{0.78}{\varphi(P) \log^2 y}
\end{aligned} \tag{29}$$

and therefore, recalling that $x_1 = \max\{\exp \max\{l, \exp 13.3\}, 10s_0(l-1) + 1\}$,

$$\begin{aligned}
V(P(z)) &= \prod_{r < z} \left(1 - \frac{\rho(r)}{r}\right) \leq \prod_{r < z} \left(1 - \frac{1}{r}\right)^{\rho(r)} \\
&= \prod_{r < z} \left(1 - \frac{1}{r}\right) \prod_{r < z, r \in U} \left(1 - \frac{1}{r}\right)^{l-1} \\
&< \frac{e^{-\gamma} \log x_1}{\log^{1-\kappa} y \log^{1+\kappa} z} \left(1 + \frac{1}{\log^2 z}\right) \\
&\quad \times \exp\left(\frac{s_0(l-1)}{x_1-1} + \frac{0.78}{\log^2 x_1} + \frac{0.78\kappa}{\log^2 y}\right) \\
&< \frac{e^{0.11-\gamma} \log x_1}{\log^{1-\kappa} y \log^{1+\kappa} z} < \frac{B_1 v^{1+\kappa}}{\log^{1-\kappa} y \log^{1+\kappa} x}.
\end{aligned} \tag{30}$$

Since $B(z) < 2.01 \leq v/2$ by (28), we have $\psi_0(v) = \xi(v)$ and therefore

$$\frac{V(P(z))}{\psi_0(v)} \leq \frac{2B_1 v^{1+\kappa} x}{\psi_0(v) \log^{1-\kappa} y \log^{1+\kappa} x} \leq \frac{2B_1 v^{1+\kappa}}{\xi(v) \log^{1-\kappa} y \log^{1+\kappa} x}. \tag{31}$$

The treatment of the remaining case $z < y$ is simpler. Similarly to the first case, we have

$$\sum_{r \leq z} \frac{\rho(r) \log r}{r} \leq \sum_{r \leq z} \frac{\log r}{r} + \sum_{r \leq z, r \equiv 1 \pmod{l}} \frac{(l-1) \log r}{r} < 2.01 \log z. \tag{32}$$

and

$$\begin{aligned}
V(P(z)) &\leq \prod_{r < z} \left(1 - \frac{1}{r}\right) \prod_{r < z, r \in U} \left(1 - \frac{1}{r}\right)^{l-1} \\
&< \frac{e^{-\gamma} \log x_1}{\log^2 z} \left(1 + \frac{1}{\log z}\right) \exp\left(\frac{s_0(l-1)}{x_1-1} + \frac{0.78}{\log^2 x_1}\right) \\
&< \frac{e^{0.11-\gamma} \log x_1}{\log^2 z} < \frac{B_1 v^2}{\log^2 x}.
\end{aligned} \tag{33}$$

By (32), we have $B(z) \leq 2.01 < v/2$ and therefore, similarly to the first case, (33) gives

$$\frac{V(P(z))}{\psi_0(v)} \leq \frac{2B_1 v^2 x}{\xi(v) \log^2 x}. \tag{34}$$

Now, with the aid of (26), the lemma easily follows from inequalities (31) and (34). \square

Now we shall prove Theorem 1.1. Let z be a real number $\geq \max\{x_2, 2(d+1)s\}$ and assume that N has no prime factor less than z .

Since $\prod_{i=1}^s h(p_i^{\alpha_i}) \leq (z/(z-1))^s$, we obtain

$$\prod_{j=1}^t h(q_j^{2\beta_j}) \geq \frac{n}{d} \times \left(\frac{z-1}{z}\right)^s > \sqrt{\frac{n}{d}}. \quad (35)$$

Let $d_l = \prod_q q/(q-1)$, where q runs all primes in Q_l . It follows from (35) that $\prod_{l \in \mathcal{P}} d_l$ must be $\geq \sqrt{\frac{n}{d}}$. Hence we have that some $d_l \geq \delta_1 = \left(\frac{n}{d}\right)^{1/2\#\mathcal{P}}$.

Let $\kappa = (l-1)/\varphi(P)$. Since N has no prime factor less than z , Lemma 4.1 gives, taking $v = 10.49$,

$$\begin{aligned} \log \delta_1 &\leq \sum_{p \geq z, p \in S} \frac{1}{p} \leq \int_z^\infty \frac{\pi_S(t)}{t^2} dt \\ &< \int_z^{z^v} \frac{2B_1 v^2}{\xi(v)t \log^2 t} dt + \int_{z^v}^\infty \frac{2B_1 v^{1+\kappa}}{\xi(v)t \log^{1+\kappa} t \log^{1-\kappa} x_2} dt \\ &= \frac{2B_1}{\xi(v) \log z} \left(v^2 \left(1 - \frac{1}{v}\right) + \frac{v}{\kappa} \right) < \frac{\log x_1}{\log z} \left(\frac{31.6}{\kappa} + 299.3 \right). \end{aligned} \quad (36)$$

Hence we obtain

$$\log z < \frac{(31.6\varphi(P)/(l-1) + 299.3) \log x_1}{\log \delta_1} \leq \frac{(63.2\varphi(P) + 598.6(l-1))\#\mathcal{P} \log x_1}{(l-1) \log \frac{n}{d}}. \quad (37)$$

This proves Theorem 1.1.

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